Characterization of a 3DOF aeroelastic system with freeplay and aerodynamic nonlinearities – Part I: Higher-order spectra

Michael Candon a,⁎, Robert Carrese a, Hideaki Ogawa a, Pier Marzocca a, Carl Mouser b, Oleg Levinski b, Walter A. Silva c

a School of Engineering (Aerospace and Aviation), RMIT University, Melbourne, Victoria, Australia
b Defence Science and Technology Group, Fishermans Bend, Victoria, Australia
c NASA Langley Research Center, Hampton, Virginia, USA

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Abstract
The identification of nonlinear systems in aeroelasticity poses a significant challenge for practitioners, often hampered by the complex nature of aeroelastic response data which may contain multiple forms of nonlinearity. Characterizing and quantifying nonlinearities is further hampered when the response is obtained at a location which is away from the nonlinear source and/or the response is contaminated by noise. In the present paper, a three-degree-of-freedom airfoil with a freeplay nonlinearity located in the control surface and exposed to transonic flow is investigated. In this Part I paper the main form of analysis is via higher-order spectra techniques to unveil features of the nonlinear mechanism which result from i) structural nonlinearities (freeplay) in isolation and ii) freeplay with Euler derived nonlinear inviscid aerodynamic phenomena (transition between Tijdeman Type-A and Type-B shock motion). It is shown that the control surface structural freeplay nonlinearity is characterized by strong cubic phase-coupling between linear and nonlinear modes. On the other hand, nonlinear inviscid flow phenomena are shown to be characterized by quadratic phase-coupling between linear and nonlinear modular modes, the strength of which is related to the strength of the aerodynamic nonlinearity (amplitude of the shock motion). The nonlinear inviscid flow phenomena do not appear to affect the identification of the freeplay nonlinearity. Conjectures are made which address the transition between aperiodic, quasi-periodic and periodic behavior (pre-flutter), further physical support towards these conjectures is provided in Part II [1]. The limitations of the higher-order spectra approach are assessed, in particular, the analysis demonstrates the difficulty in extracting natural frequencies with this approach.

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1. Introduction
Within the body of knowledge surrounding aeroelasticity for aircraft it is well documented in the literature that both aerodynamic and structural nonlinearity can induce undesirable nonlinear aeroelastic phenomena. In particular, limit cycle
<table>
<thead>
<tr>
<th>Nomenclature</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>pitching displacement</td>
</tr>
<tr>
<td>$\beta$</td>
<td>control surface displacement</td>
</tr>
<tr>
<td>$\beta_s$</td>
<td>control surface freeplay margin</td>
</tr>
<tr>
<td>$\mu$</td>
<td>structural-to-fluid mass ratio</td>
</tr>
<tr>
<td>$\omega_\alpha$</td>
<td>pitch natural frequency (coupled)</td>
</tr>
<tr>
<td>$\omega_\beta$</td>
<td>control surface natural frequency (coupled)</td>
</tr>
<tr>
<td>$\omega_h$</td>
<td>plunge natural frequency (coupled)</td>
</tr>
<tr>
<td>$\omega_{NL}$</td>
<td>nonlinear mode</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>phase</td>
</tr>
<tr>
<td>$\tau$</td>
<td>tricoherence</td>
</tr>
<tr>
<td>$a$</td>
<td>normalized distance from the elastic axis to mid-chord</td>
</tr>
<tr>
<td>$A_r$</td>
<td>asymptotic range calculated for the grid convergence index</td>
</tr>
<tr>
<td>$b$</td>
<td>semi-chord</td>
</tr>
<tr>
<td>$b$</td>
<td>bicoherence</td>
</tr>
<tr>
<td>$B$</td>
<td>bispectrum</td>
</tr>
<tr>
<td>$C_l$</td>
<td>lift coefficient</td>
</tr>
<tr>
<td>$C_{m_x}$</td>
<td>moment coefficient of the airfoil</td>
</tr>
<tr>
<td>$C_{m_\beta}$</td>
<td>moment coefficient of the control surface</td>
</tr>
<tr>
<td>$C$</td>
<td>damping matrix</td>
</tr>
<tr>
<td>$d$</td>
<td>normalized distance from the mid-chord to hinge</td>
</tr>
<tr>
<td>$f$</td>
<td>frequency</td>
</tr>
<tr>
<td>$F$</td>
<td>nonlinear internal forces</td>
</tr>
<tr>
<td>$F$</td>
<td>fourier transform</td>
</tr>
<tr>
<td>$F_s$</td>
<td>safety factor used in the grid convergence index calculations</td>
</tr>
<tr>
<td>$g$</td>
<td>damping ratio</td>
</tr>
<tr>
<td>$GCI_{12,23}$</td>
<td>grid convergence index between refinement levels 1 and 2, and, 2 and 3 respectively</td>
</tr>
<tr>
<td>$h$</td>
<td>plunging displacement</td>
</tr>
<tr>
<td>$l_x$</td>
<td>non-dimensional airfoil moment of inertia</td>
</tr>
<tr>
<td>$l_\beta$</td>
<td>non-dimensional control surface moment of inertia</td>
</tr>
<tr>
<td>$j_{1,2,3}$</td>
<td>general terms used for the physical solutions to each grid in the grid convergence index</td>
</tr>
<tr>
<td>$k_h$</td>
<td>plunge longitudinal stiffness</td>
</tr>
<tr>
<td>$k_x$</td>
<td>pitch torsional stiffness</td>
</tr>
<tr>
<td>$k_\beta$</td>
<td>control surface torsional stiffness</td>
</tr>
<tr>
<td>$K$</td>
<td>stiffness matrix</td>
</tr>
<tr>
<td>$L$</td>
<td>lift</td>
</tr>
<tr>
<td>$m$</td>
<td>airfoil mass per-unit-span</td>
</tr>
<tr>
<td>$M$</td>
<td>higher-order spectra number of data segments</td>
</tr>
<tr>
<td>$M_x$</td>
<td>airfoil pitching moment</td>
</tr>
<tr>
<td>$M_\beta$</td>
<td>control surface pitching moment</td>
</tr>
<tr>
<td>$M_\infty$</td>
<td>freestream Mach number</td>
</tr>
<tr>
<td>$M$</td>
<td>mass matrix</td>
</tr>
<tr>
<td>$N$</td>
<td>total number of data points</td>
</tr>
<tr>
<td>$q_\infty$</td>
<td>freestream dynamic pressure</td>
</tr>
<tr>
<td>$r_s$</td>
<td>normalized radius of gyration about the elastic axis</td>
</tr>
<tr>
<td>$r_\beta$</td>
<td>normalized radius of gyration of the control surface about the control surface hinge axis</td>
</tr>
<tr>
<td>$P$</td>
<td>order of convergence for the grid convergence index</td>
</tr>
<tr>
<td>$S_x$</td>
<td>non-dimensional airfoil static moment</td>
</tr>
<tr>
<td>$S_\beta$</td>
<td>non-dimensional control surface static moment</td>
</tr>
<tr>
<td>$SP_{NL}$</td>
<td>general nonlinear spectra term</td>
</tr>
<tr>
<td>$SP_{NL,norm}$</td>
<td>general normalized nonlinear spectra term</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>time-step size</td>
</tr>
<tr>
<td>$T$</td>
<td>trispectrum</td>
</tr>
<tr>
<td>$u$</td>
<td>displacement vector</td>
</tr>
<tr>
<td>$\dot{u}$</td>
<td>velocity vector (first derivative of $u$ with respect to time)</td>
</tr>
<tr>
<td>$\ddot{u}$</td>
<td>acceleration vector (second derivative of $u$ with respect to time)</td>
</tr>
<tr>
<td>$V^*$</td>
<td>velocity index</td>
</tr>
<tr>
<td>$x_{aft}$</td>
<td>maximum aft location of the shock</td>
</tr>
<tr>
<td>$x_{for}$</td>
<td>maximum forward location of the shock</td>
</tr>
</tbody>
</table>
oscillations (LCO) can compromise the structural integrity of an airframe causing considerable permanent damage and in some cases fatigue or catastrophic failure. LCOs are induced by the presence of structural and aerodynamic nonlinearity within an aeroelastic system. Aerodynamic nonlinearities generally result from transonic flow phenomena, i.e., shock wave-boundary layer interactions and shock induced separation, high angle of attack dynamic stall operations or, within critical regions of the flight regime, transonic buffet. Structural nonlinearity can be either distributed or concentrated. Distributed nonlinearities will affect the entire wing (i.e. nonlinear material properties or stiffening due to large deformation), whereas concentrated nonlinearities are localized to regions of the airframe, e.g., at locations where there are structural loads due to externally mounted pylons with scientific payload and ordnance, freeplay, backlash providing nonlinearity and friction [2]. For a given aeroelastic response which appears to exhibit nonlinear aeroelastic phenomena, nonlinear system identification techniques are employed with the aim of detecting, characterizing and quantifying the type of nonlinearity. Identifying such systems can become an arduous task if the system contains multiple types of nonlinearity, or the type of nonlinearity is not distinguishable due to high-amplitude dynamic oscillations or contaminated by a low signal-to-noise ratio. Considering this, there is a requirement for the development of a robust system identification framework capable of detecting and characterizing single or multiple types of nonlinearity in complex noisy aeroelastic signals. Furthermore, there is a requirement to improve our comprehension of how different types of nonlinearity drive nonlinear aeroelastic phenomena, namely chaotic response and limit cycle oscillations.

Generally, freeplay is considered to be inherited from loosened mechanical linkages between the main wing and control surfaces, main wing and store link system or within an all-movable horizontal tail. Current design criteria for freeplay induced LCO are outdated and based on an elementary comprehension of the phenomena [3], hence, it is necessary to improve our understanding of freeplay induced LCO characteristics. Control surface freeplay is the most common form of freeplay and has been found to induce LCO in various civilian and defense based assets. However, as to be expected these occurrences are not well documented in the public domain. Loosened mechanical linkages can also lead to cubic stiffening within a linkage (rather than a freeplay, which is representative of a dead zone in the control actuation). Authoritative studies on freeplay nonlinearity for typical sections are provided for subsonic flow in [4–6]. While the study of n-DOF aeroelastic systems is predominantly conducted for subsonic flow regimes, progress has also been made in transonic and low-supersonic flow. Kousen and Bendicksen [7], Kim and Lee [8], Dowell et al. [9] demonstrated that in transonic and low-supersonic flow the flutter speed of n-DOF 2D airfoils with freeplay reduces significantly when compared to the linear case. Further it is shown that the “transonic dip” remains, however, with reduced flutter speeds. Dowell et al. [9] also show that at transonic Mach numbers a rapid change in flutter mode occurs. Interestingly, it is shown by Park et al. [10] that in transonic flow the catastrophic flutter boundary increases as the freeplay margin increases. Morton and Beran [11] and Dimitrijević [12] present bifurcation analysis of a similar system demonstrating that super- and subcritical bifurcations exist in the presence of freeplay nonlinearity. In a selection of recent publications, the authors investigate the limit cycle behavior of 2DOF aeroelastic systems with structural freeplay and time-marching computational fluid dynamics (CFD) to represent the unsteady aerodynamic forces, comparing the results to those obtained using traditional and reduced-order aerodynamic models [13–15]. Yang et al. [13] demonstrate that the LCO solutions obtained using linear aerodynamics agree well with those obtained using the nonlinear aerodynamics which suggests that the effects of aerodynamic nonlinearity are weak. He et al. [15] describe the effects of the aerodynamic nonlinearity on the system as the amplitude of the limit cycle grows. It is shown that low amplitude LCO behavior is characterized by single-degree-of-freedom (SDOF) flutter with negligible impact from the nonlinear aerodynamics. For moderate amplitude LCO behavior the response is dominated by the structural freeplay nonlinearity and the weak nonlinear aerodynamic forces can be successfully represented by a reduced order model (ROM). For large amplitude limit cycle responses the aeroelastic response is dominated by the nonlinear aerodynamic forces.

In inviscid transonic regimes complex unsteady shock motion may occur as a result of the wing dynamic structural motion. The relationship between the motion of the shock and the dynamic structural motion of the wing is generally linear. However, as the motion increases in amplitude it can shift from Tijdeman Type-A to Type-B shock motion [16], which is characterized by disappearance and reappearance of a shock wave. This discontinuity of the pressure distribution across the lifting surface can introduce a dynamic nonlinearity [17]. Nonlinear transonic aerodynamic phenomena for typical sections have been studied extensively, with some examples included in [18–22].

Higher-order spectra (HOS) analysis is a valuable tool for the analysis of nonlinear aeroelastic systems. The superiority of HOS in contrast to traditional linear methods, such as the power spectrum, comes from the ability of the higher-order statistics to provide insight into the presence, strength and functional form of nonlinearities, whereas the power spectrum is only able to define second-order statistics, therefore, can only rigorously unveil physics associated with linear phenomena [23,24]. It is also worth noting the duality between HOS and Volterra functional series’ kernels, where HOS can be represented as the Fourier transform of the Volterra series, i.e., the Volterra series in the frequency-domain [25]. HOS analysis
has been applied recently to flight test, wind tunnel and numerical data to investigate the nonlinear aeroelastic aspects of various full aircraft, wing and airfoil configurations [26–30]. It is shown that HOS analysis is able to detect the presence of nonlinearity and provide insight into the transition from linear to nonlinear behavior. For an in-depth literature survey on the application of HOS to nonlinear aeroelastic systems refer to [31].

In a recent study by the authors [31], HOS are utilized to characterize aeroelastic predictions via a 2DOF aeroelastic system with control surface freeplay and aerodynamic nonlinearity. Evidence is provided to suggest that the freeplay can be characterized by a cubic process while the nonlinear shock–motion appears to be characterized a quadratic process, however, this observation is not fully verified. The present research aims to build upon this work by characterizing the higher-order modular frequency interactions which are identified within the nonlinear aeroelastic response of a three-degree-of-freedom (3DOF) airfoil system with control surface freeplay and inviscid aerodynamic nonlinearities. In order to isolate the freeplay from other forms of nonlinearity, specifically in this case nonlinear aerodynamic contributions from transonic flow, a linearized aerodynamic model is considered where the linearization is performed about a high-fidelity Euler steady-state solution. Nonlinear inviscid aerodynamics is then included and quantified by resolving the generalized aerodynamic forces extracted from the high-fidelity Euler simulations at each time-step, inherently accounting for unsteady Euler steady-state solution. The novel elements of this work are: i) the use of HOS to characterize and identify the nonlinear mechanism associated with the combined effects of structural freeplay and nonlinear inviscid aerodynamics, ii) new understanding of the underlying physics of the nonlinear system by analyzing the development and change of the linear and nonlinear spectral content as the system transitions from low-amplitude aperiodic response to LCO and iii) novel physical insights towards the interaction between structural freeplay and aerodynamic nonlinearity. Ultimately, the objective of point i) is to be a step towards an automated nonlinearity diagnosis framework which would reduce the time and costs associated with aircraft maintenance. More precisely, when the response provided by the aircraft’s sensory systems (e.g., strain gauges or accelerometers) appears to be nonlinear (e.g., LCO or chaotic response), the framework aims to give the practitioner an indication as to the type of nonlinearity and where it is located spatially. Thus, reducing the requirement for routine maintenance or physical inspection/flight testing when the assets sensory systems exhibit a nonlinear response.

The remainder of the paper is constructed as follows: Section 2 provides details of the airfoil model and numerical framework. Section 3 briefly defines the HOS approach and presents HOS validation/sensitivity studies. Section 4 presents the results and discussion. Finally, Section 5 summarizes the study and prospects for future work.

2. Computational framework

2.1. Computational model

In the present research a typical section of NACA0012 profile is considered supported by plunging and torsional springs, further equipped with a trailing-edge control surface, connected via a torsional spring to the main section as presented in Fig. 1. The plunging and pitching motions are denoted by \( h \) and \( \alpha \), and the control surface motion is denoted by \( \beta \). The plunge and the pitch are measured at the elastic axis and the control surface deflection is measured from the hinge line. The structural configuration is described by the parameters shown in Table 1. The normalized parameters are according to the semi-span, \( b \).

![Fig. 1. a) NACA0012 section with control surface and b) flap stiffness as a function of flap rotation.](image-url)
2.2. Numerical approach

The equation of motion for the generalized nonlinear aeroelastic system in discrete (nodal) coordinates for a given free-stream Mach number \( M_\infty \) and Reynolds number \( Re_\infty \) is given as

\[
[M]\{\ddot{\mathbf{u}}\} + [C]\{\dot{\mathbf{u}}\} + [K]\{\mathbf{u}\} + \{\mathbf{F}(\mathbf{u})\} = q_\infty \{\mathbf{Q}\}(M_\infty, Re_\infty)
\]  

(1)

where \([M], [C]\) and \([K]\) are the structural mass, damping and stiffness matrices, and \(\{\mathbf{u}\} = \{u_1, u_2, \ldots, u_N\}^T\) is the displacement vector of \(N\) degrees-of-freedom, with the usual notation that \(\{\ddot{\mathbf{u}}\}\) and \(\{\dot{\mathbf{u}}\}\) are the first and second derivatives of \(\{\mathbf{u}\}\) respectively, with respect to the time, \(t\). Nonlinear internal forces are represented by the term \(\{\mathbf{F}(\mathbf{u})\}\). \(\{\mathbf{Q}\}\) is solely due to structural motion according to

\[
\mathbf{Q} = \mathbf{Q}(\mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}, t)
\]  

(2)

In the present work an undamped aeroelastic system is considered, hence, according to the Lagrange equation, the linear equations of motion for the 3DOF NACA0012 profile are written in a matrix form as

\[
\begin{bmatrix}
m & S_x & S_y & \hbar \\
S_x & I_x & I_y + b(d - a)S_y & \dot{\beta} \\
S_y & I_y + b(d - a)S_y & I_y & \ddot{\beta}
\end{bmatrix}
\begin{bmatrix}
\ddot{h} \\
\ddot{z} \\
\ddot{\beta}
\end{bmatrix}
+ \begin{bmatrix}
k_h & 0 & 0 \\
0 & k_z & 0 \\
0 & 0 & k_\beta
\end{bmatrix}
\begin{bmatrix}
h \\
z \\
\beta
\end{bmatrix}
= \begin{bmatrix}
L \\
M_x \\
M_\beta
\end{bmatrix}
\]

where \(S_x = mbx_x\) and \(S_y = mbx_y\) are the static moments, and \(I_x = mb^2r_x^2\) and \(I_y = mb^2r_y^2\) are the moments of inertia. The aerodynamic forces are defined by the lift \(L\) and pitching moment \(M_x\) of the wing section about the elastic axis, and pitching moment coefficient of the control surface about the hinge point \(M_\beta\). \(k_h, k_z,\) and \(k_\beta\) are the linear stiffness values for the longitudinal, torsional and hinge springs, respectively.

Now the concentrated structural nonlinearity can be included by setting \(k_\beta\) in the above equation to zero and including \(\mathbf{F}(\mathbf{u})\) as per Eq. (1). \(\mathbf{F}(\mathbf{u})\) is defined by

\[
\mathbf{F}(\mathbf{u}) = \begin{bmatrix}
0 \\
0 \\
F(\beta)
\end{bmatrix}
\]

where

\[
F(\beta) = \begin{cases}
0 & \text{if } -\beta_s < \beta \leq \beta_s \\
k_\beta(\beta - \beta_s), & \text{if } \beta > \beta_s \\
k_\beta(\beta + \beta_s), & \text{if } \beta < -\beta_s
\end{cases}
\]

This nonlinear aeroelastic equation of motion is to be solved via two approaches in the present research: a linearized state-space approach to isolate the freeplay nonlinearity and full nonlinear time-integration to include the effects of freeplay and aerodynamic nonlinearity.

2.2.1. Linearized state-space approach

The linearized state-space approach considers the fundamental assumption that the generalized aerodynamic forces are linear and a linearized relationship of the aerodynamic forces with respect to the structural motion is imposed.
By neglecting the unsteady terms in Eq. (2), the aerodynamic forces can be provided in transfer function form for solution of Eq. (1) in the frequency-domain, such that:

\[
Q(j\omega) = |H(j\omega)||u(j\omega)| \quad (3)
\]

where \(\omega\) is the oscillation frequency. An equivalent expression is given in the time-domain where Eq. (3) is recast in terms of the convolution integral

\[
Q(\tau) = \int_{0}^{\infty} [H(t-\tau)||u(\tau)||] dt \quad (4)
\]

where \(H\) is the force response to an arbitrary impulse in \(u\).

To generate the aerodynamic transfer function in the present research a blended step input signal is used to excite the aerodynamic system for an entire range of reduced frequencies \(k\). This is computed using a high-fidelity Euler based solver in ANSYS Fluent [32]. Assuming a small perturbation about the system in equilibrium, the generalized (linear) aeroelastic system can be recast in the Laplace domain, where the aerodynamic transfer function matrices are transcendental functions of the form \(H(jk, M_\infty, Re_\infty)\). The rational function approximation (RFA) of Roger [33] is used to generate a state-space approximation of the aerodynamic model. In this framework freeplay nonlinearities and cubic stiffness effects are easily implemented by modifying the stiffness contributions. The state-space system is then solved using a fourth-order Runge-Kutta approach providing highly efficient nonlinear aeroelastic response data in discrete time.

### 2.2.2. Full nonlinear time-integration

Full nonlinear time-integration is utilized to include the aerodynamic nonlinearity. To capture the effects of aerodynamic nonlinearities the generalized aerodynamic forces are solved with time-accurate fully nonlinear coupling between the structural model and the unsteady aerodynamic loading. Euler-based CFD is used to capture the inviscid aerodynamic loading. Here the nonlinear aeroelastic equations of motion are solved directly in ANSYS Fluent [32], via a fourth-order Runge-Kutta scheme. The nonlinear structural model is embedded in ANSYS Fluent via a user defined function (UDF) and solved via a tightly coupled implicit scheme. A maximum of 20 CFD iterations are allowed per time-step with the CFD solution generally converging within 5–10 iterations. The Euler equations for the transient flowfields are solved via a coupled pressure-based solver with second-order upwind spatial accuracy. The convergence criteria are set to \(1 \times 10^{-3}\) for the scaled residuals at each time step. An O-type structured grid of \((120 \times 46)\) elements is used as presented in Fig. 2. A dynamic mesh with a smoothing method is utilized to facilitate the motion of the airfoil. The transient time-step is set to \(1 \times 10^{-3}\) s. The transient simulations are initialized from a converged steady-state solution.

### 3. Higher-order spectra

This study employs bispectral and trispectral density analysis to identify quadratic and cubic nonlinearities respectively within the aeroelastic system, i.e., more specifically the nonlinearities introduced through freeplay and aerodynamic nonlinearity. A generalized term for the nonlinear spectral density can be defined by
where the block length $M$ is the number of data segments to be considered, $SP_{NL}$ can be the bispectrum $B$ ($n = 2$) or trispectrum $T$ ($n = 3$), $Y(f)$ is the Fourier transform of the time-series $\mathcal{F}(y(t))$, and $Y^*(f)$ is the complex conjugate of $Y(f)$. The bispectrum $B(f_1, f_2)$ identifies quadratic nonlinearities within the aeroelastic system by estimating third-order moments in the frequency domain. The trispectrum at any bifrequency $(f_1, f_2)$ measures the level of interaction between the two frequencies $f_1$ and $f_2$. The interactions are a result of quadratic phase-coupling between the frequency components and hence the bispectrum detects the presence of quadratic nonlinearities. Similarly, the trispectrum $T(f_1, f_2, f_3)$ identifies cubic nonlinearities within the aeroelastic system by estimating fourth-order moments in the frequency domain. The trispectrum at any trifrequency $(f_1, f_2, f_3)$ measures the level of interaction between the three frequencies $f_1, f_2$ and $f_3$. The interactions are a result of cubic phase-coupling between the frequency components and hence the trispectrum detects the presence of cubic nonlinearities.

To give a better understanding of the nature of the identification of phase-coupling which occurs via the nonlinear spectra, one can consider the definition as

$$SP_{NL}(f_1, \ldots, f_n) = \frac{1}{M} \sum_{i=1}^{M} Y_i(f_1) \cdots Y_i(f_n) Y_i^*(f_1 + \cdots + f_n)$$

(5)

Although particular attention must be paid to the number of data points $N$ and block length $M$ in estimating the higher-order spectral content of a system, the recommendation by Dalle Molle and Hinch [35] is highly impractical in most cases; for example, to estimate the trispectrum (fourth-order cumulant) using a 512-point FFT (block length of 512 points) the required number of data points would be 134,217,728. As this number of data points is formidable, the authors have conducted multiple time-step and block length convergence studies, as an example included in [31]. It is shown that to give an indication as to whether the system is defined by a linear, quadratic and/or cubic process (in normalized form) while minimizing computational cost for a SDOF mass-spring-damper (MSD) system a minimum of $2^{17}$ data points is sufficient, no less than $2^{15}$ should be used as a minimum. This is to give an indication of the bi-/tricoherence value to within 0.1 where the entire-bi/tricoherence spectrum is defined between 0 and 1. For more precise HOS estimations the recommendations of Dalle Molle and Hinch [35] should be considered.

This result cannot be considered as a general rule and rigorous convergence studies should be conducted prior to gaining confidence in the HOS estimates on a case-by-case basis. For the nonlinear aeroelastic system presented in this manuscript a convergence study is conducted as presented in A.

4. Results

This section initially presents a numerical validation of the 3DOF aeroelastic model, followed by linear flutter solutions which identify the divergent flutter point and describe the behavior of the dynamic system as it approaches flutter. Three forms of nonlinear analysis are presented using linear and nonlinear spectral approaches, including:
The system in vacuum driven by Gaussian forcing functions with cubic stiffening in the control surface and a quadratic nonlinearity (in the form of quadratic stiffening) imposed on the main surface.

- The system with control surface structural freeplay driven by linearized aerodynamics.
- The system with control surface structural freeplay driven by Euler derived nonlinear inviscid aerodynamics.

All simulations are conducted at a fixed freestream Mach number of $M_1 = 0.8$ with the reduced velocity $V' = V_\infty/\sqrt{\mu \omega_b}b$ varied, where $\mu = m/\pi \rho_\infty b^2$ is the structural-to-fluid mass ratio.

### 4.1. Validation and linear flutter analysis

Initially linear flutter analysis is conducted with three objectives: i) to provide numerical validation for the solution methodologies against Dowell et al. [9], ii) to identify the linear flutter boundary of the system at $M_1 = 0.8$ and iii) to understand how the natural modal frequencies vary with speed.

#### 4.1.1. Numerical validation of the linear system

To validate the two solution methodologies, namely the linear structural system with linearized and unsteady Euler aerodynamics, the linear flutter results obtained here are compared with the numerical linear flutter results of Dowell et al. [9] over a range of Mach numbers. For the linearized solver $V - g$ plots are used to determine the linear flutter boundary and the mode by which linear flutter occurs over the Mach numbers of interest. For the full nonlinear solver, the reduced velocity is increased incrementally and the time-histories are analyzed until it is observed that the oscillations are no longer converging, which is repeated over the Mach numbers of interest. It can be seen in Fig. 3 that as $M_\infty$ increases and the strength of shock wave in the transonic flow increases, the discrepancies between the work of Dowell et al. [9] and the current predictions reduce. As the remainder of the aerodynamic modeling in this paper (Sections 4.3, 4.4) considers a constant Mach number of $M_\infty = 0.8$ (where there is good agreement between the three numerical models) the two aerodynamic solvers can be used in confidence.

#### 4.1.2. Linear flutter analysis

The $V' - g$ plot in Fig. 4 a) demonstrates that linear flutter occurs at $V' = 0.66$ via the plunge mode and Fig. 4 b) indicates that at $V' = 0.66$ the flutter frequency (natural plunging mode $\omega_n$) is at $\omega_n = 12.2$ Hz. Although not shown here, both the bicoherence and tricoherence estimates at the neutral point indicate negligible nonlinear spectra ($< 0.01$) in all modes verifying that the linearized solver has indeed generated linear results. The flutter mechanism is investigated in further detail in Section 4.4.

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1 The parameters used for validation are taken from Dowell et al. [9]. These are not consistent with the parameters used for the analysis in the remainder of this paper. The authors modified the structural parameters for the analysis in the present paper to provide aeroelastic response signals that, as speed increases, would display multiple distinguishable features (i.e., aperiodic, quasi-periodic and periodic responses). Since the work of Dowell et al. [9] was not validated against experimental data, the authors proceed with a numerical based validation.
4.2. 3DOF airfoil in vacuum with gaussian input

The purpose of this section is three-fold; i) to provide the reader, in a simplified manner, an example of how HOS can be used to detect quadratic and/or cubic phase-coupling between the structural modes of a system in the presence of nonlinearities, hereby indicating whether the type of nonlinearity acting is quadratic or cubic, ii) to show that the strength of the bi/tricoherence can indicate where in the system the nonlinearity is acting, and iii) to verify that the HOS code is providing reasonable and valid results.

Here, the aeroelastic equations of motion are modified by removing the aerodynamic component which is replaced by a Gaussian white noise forcing function (low pass filtered through 100 Hz). It is to be shown in Sections 4.3 and 4.4 that the aerodynamic nonlinearity can be characterized by a quadratic process and the structural freeplay by a cubic process. As a result the nonlinearities included in this section are:

- Quadratic stiffness in the pitching DOF of the main section to simulate the aerodynamically generated quadratic nonlinearity.
- Cubic stiffening in the control surface to represent the structural freeplay nonlinearity. The reason the cubic hardening term is included (rather than a freeplay dead-zone) is to simplify this example case and provide a clean set of nonlinear interactions for the reader to follow.

**Fig. 4.** a) Damping ratio as a function of velocity index \((V - g)\) and b) modal frequency as a function of velocity index \((V - \omega)\) plots.
The nonlinear equations of motion are now defined according to

\[
\begin{bmatrix}
  m & S_x & S_y & S_z \\
  S_x & I_x & I_y + b(d - a)S_y & S_z \\
  S_z & I_y + b(d - a)S_x & I_y & S_z \\
  S_y & I_y & S_z & I_y
\end{bmatrix}
\begin{bmatrix}
  \dot{h} \\
  \dot{\alpha} \\
  \dot{\beta} \\
  \dot{\gamma}
\end{bmatrix}
+\begin{bmatrix}
  k_h & 0 & 0 \\
  0 & k_\alpha & 0 \\
  0 & 0 & k_\beta
\end{bmatrix}
\begin{bmatrix}
  h \\
  \alpha \\
  \beta \\
  \gamma
\end{bmatrix}
+ F[u] = \begin{bmatrix}
  X_h.Gauss \\
  X_\alpha.Gauss \\
  X_\beta.Gauss
\end{bmatrix}
\]

Where \( F[u] \) is defined according to

\[
F[u] = \begin{cases}
  0 & \text{if } u = F_{\text{quad}} \\
  -F_{\text{quad}} & \text{if } u = F_{\text{cubic}}
\end{cases}
\]

Herein \( X_h.Gauss, X_\alpha.Gauss \) and \( X_\beta.Gauss \) represent the Gaussian noise forcing functions in the respective DOFs and \( F_{\text{quad}} = k_{\text{quad}}x^2 \) and \( F_{\text{cubic}} = k_{\text{cubic}}x^3 \) represent the artificial aerodynamic nonlinearity and cubic control surface hinge stiffening respectively.

4.2.1. Example: higher-order statistics

Fig. 5 presents the PSD plots of the Gaussian driven system. It can be seen that the dominant frequency in each DOF is driven by its natural frequency, i.e., \( \omega_h \) in plunge, \( \omega_\alpha \) in pitch and \( \omega_\beta \) in the control surface hinge (as to be expected). The frequencies of each of the other modes can also be detected at each location, however, at a lower energy. A range of nonlinear interactions can also be observed which are a result of quadratic and cubic phase-coupling between the linear aeroelastic modes.

Figs. 6–8 present the bicoherence (quadratic phase-coupling) and tricoherence (cubic phase-coupling) estimates in each DOF for the Gaussian driven system. Figs. 6a), 7a) and 8a) indicate that the artificial aerodynamic forcing function \( F_{\text{quad}} \) leads to moderate – strong quadratic phase-coupling at all locations. In plunge this is via the bi-interaction of \( \omega_h \). Weak quadratic phase-coupling is also evident via the bi-interaction of \( \omega_x \) and where \( \omega_x \) interacts with \( \omega_y - \omega_z \). In pitch the quadratic phase-coupling is strongest (near unity), which is expected as the artificial quadratic aerodynamic forcing function is applied in the pitching DOF. The strongest interaction here is where \( \omega_x \) interacts with \( \omega_y - \omega_z \) indicating that the modes are tightly coupled. There is also weak quadratic phase-coupling via the bi-interaction of \( \omega_x \) and \( \omega_y \). In the control surface there is strong quadratic phase-coupling via the bi-interaction of \( \omega_x \) with no other interactions evident, likely due to the spatial difference between the pitch and control surface hinges.

Figs. 6b), 7b) and 8b) demonstrate that the cubic nonlinearity in the control surface is strong while in the other two modes the cubic is weak. Again, this is expected as the cubic stiffness is in the control surface and there is a spatial difference between the control surface hinge and the connection point of the other two modes. In plunge the weak cubic phase-coupling is the strongest via the tri-interaction of \( \omega_h \) and where \( \omega_h \) interacts with the bi-interacting \( \omega_x \). There is very weak cubic phase-coupling where \( \omega_x \) interacts with the bi-interacting \( \omega_h \). In pitch the strongest cubic phase-coupling is where the bi-interacting \( \omega_x \) interacts with \( \omega_h \). There is weaker coupling via the tri-interaction of \( \omega_x \) and \( \omega_y \) and where the bi-interacting \( \omega_x \) interacts with \( \omega_z \). Again, the interactions in pitch and plunge emphasize the tight coupling between the modes. In plunge there is strong cubic phase-coupling via the tri-interaction of \( \omega_p \) and very weak cubic phase-coupling where the bi-interacting \( \omega_x \) interacts with \( \omega_z \).
4.3. Pre-limit cycle

The current section considers a structural freeplay nonlinearity in the control surface of $\beta_i = \pm 0.5^\circ$ and is driven by generalized aerodynamic forces acting upon each mode. The behavior and higher-order frequency content are investigated for speeds below the nonlinear flutter boundary (LCO boundary). Spatial and temporal convergence studies for the present and following sections are presented in B.

Fig. 9 presents a bifurcation diagram for the control hinge rotation response. Several characteristic behaviors are observed, including:

- Aperiodic response from $V^* = 0.2 - 0.29$ and $V^* = 0.38 - 0.55$
- At $V^* = 0.28$ the chaotic system shifts to a multi-amplitude periodic response in which several well defined branches can be observed before breaking down again at $V^* = 0.38$.
- A subcritical bifurcation at $V^* = 0.57$ which can be considered the nonlinear flutter boundary and is characterized by LCO of which the amplitude grows before flutter occurs at $V_f = 0.66$.

**Fig. 6.** a) Bicoherence and b) tricoherence estimations in plunge with the system in vacuum driven by Gaussian forcing functions.

**Fig. 7.** a) Bicoherence and b) tricoherence estimations in pitch with the system in vacuum driven by Gaussian forcing functions.
The following spectral analysis will investigate velocity index values which correspond to each of these observed changes in nonlinear behavior (prior to LCO), i.e., $V^* = 0.25, 0.35$ and $0.45$.

For $V^* = 0.25, V^* = 0.35$ and $V^* = 0.45$ the results indicate good agreement between the linearized and nonlinear solutions such that all modes are found to be defined by weak quadratic and strong cubic processes (Figs. 10–13), which suggests that the freeplay nonlinearity (in isolation from nonlinear aerodynamics) can be identified by a cubic process. This is to be expected as the function $x_b(b)$ (Fig. 1) is akin to a cubic form. Fig. 10 indicates that the system is characterized by the strong presence of $x_h$ and a range of nonlinear modular modes which are not harmonically related to the linear aeroelastic modes, but, are defined by interactions between them; $x_h$, $x_a$ and $x_b$. At all speeds a range of fundamental low frequency modes exist as a result of nonlinear interactions. The energy of these modes is much less than that of the dominant frequencies, albeit, identification of these modes is essential in understanding the periodicity of the response and in defining the higher frequency dominant nonlinear modes. Table 2 presents a summary of the natural frequencies and the fundamental nonlinear frequencies at each speed.

At $V^* = 0.25$ (aperiodic response) it is shown in Fig. 9 that the response is aperiodic in nature. Here, the low frequency nonlinear modes $\omega_{NL1} - \omega_{NL4}$ are formed via interactions between all three linear aeroelastic modes. Weak quadratic phase-coupling is observed (Fig. 11a) where the strongest interactions are via the bi-interactions $[(\omega_h)(2\omega_h - \omega_{NL2} = 4\omega_h - \omega_p/2) = 15.50$ Hz] and $[(\omega_h)(\omega_h + \omega_{NL3} = \omega_h + \omega_p - 2\omega_a) = 16.79$ Hz]. Here, moderate cubic nonlinearities are identified in the plunging DOF and strong cubic nonlinearities in the pitching and control hinge DOFs.
In the control hinge (Fig. 11b) strong cubic phase-coupling is observed via the tri-interaction \([\omega_h, (\omega_h)^3, (\omega_h)^3]\), moderate-strong cubic phase-coupling via the tri-interaction \([\omega_h, (\omega_h)^3, (\omega_h)^3, (\omega_h/2)^3]\) and moderate cubic-phase coupling via the tri-interaction \([\omega_h, (\omega_h)^3, (\omega_h)^3, (\omega_h/2)^3]\)

At \(V^* / C_3 = 0.35\) (periodic response) (also pertinent to the range of speeds \(0.29 < V^* / C_3 < 0.38\)) Fig. 9 indicates a behavioral change such that the system has shifted to a multi-amplitude periodic response. The initialization of the quasi-periodic behavior at \(V^* / C_3 = 0.29\) can be explained by the harmonic relation \(\omega_2 = 2\omega_h\) at this speed, hence, the tri-modal nonlinear system becomes bi-modal and inherently shifts to a greater state of order. This harmonic relation could also be described as an internal resonance (IR) phenomenon \([37,38]\), which will be discussed further in Part II \([1]\). Here, Fig. 10 indicates a strong peak at \(\omega_h/3 = 3.39\) Hz. It is speculated that the frequency lock-in that is observed at \(V^* / C_3 = 0.35\) can be explained by the relationship \(\omega_{NL2} = (\omega_h - 4\omega_h/3) = 3.28\) Hz \(\approx \omega_h/3 = 3.42\) Hz. It follows that it is preferential for the system to remain in a higher state of order by locking into \(\omega_h/3\) as opposed to returning to a state of disorder. Although not displayed here, as \(V^*\) increases to values \(> 0.35\) the value of \(\omega_{NL5}\) diverges from \(\omega_h/3\) until a critical value where the system becomes aperiodic again. That is, at this point the system once again becomes tri-modal which coincides with the breakdown of the quasi-periodic (ordered) state, and hence the system shifts back towards an aperiodic (disordered) state. The harmonic relation \(\omega_2 = 2\omega_h\) at \(V^* = 0.306\) explains the onset of the periodicity. We can conjecture that the ongoing presence of the periodic region (beyond \(V^* = 0.29\)) is due to the relationship \(\omega_{NL2} \approx \omega_h/3\), which induces a frequency lock-into \(\omega_h/3\) within the range.
of speeds $0.29 < V^* < 0.38$. A further conjecture is made towards the breakdown of the periodic region at $V^* = 0.4$ due to the divergence of the values $\omega_{NL2}$ and $\omega_h/3$. Albeit, a sound physical reasoning for the presence of the periodic region beyond onset and the breakdown of the region is not fully provided here. This is sufficiently interesting to be further investigated in Part II [1] of this paper by tracking the frequency of the nonlinear modes within the periodic region as $V^*$ increases.

At $V^* = 0.35$ weak quadratic phase-coupling is observed (Fig. 12 a) where the strongest interactions are via the self-interaction of $[(2\omega_h + \omega_h/3), (2\omega_h + \omega_h/3)]$ and at $[(\omega_h), (2\omega_h + \omega_h/3)]$. Here, moderate-strong cubic nonlinearities are identified in the plunge DOF, and strong cubic nonlinearities in the pitch and control hinge DOFs. In the control hinge (Fig. 12b), strong cubic phase-coupling can be observed via the tri-interaction $[(\omega_h), (\omega_h), (2\omega_h + \omega_h/3)]$, and moderate cubic phase-coupling via the tri-interaction of $[(2\omega_h + \omega_h/3), (2\omega_h + \omega_h/3), (2\omega_h + \omega_h/3)]$.

At $V^* = 0.45$ (aperiodic response) the response returns to being aperiodic in nature as presented in Fig. 9. This is also pertinent to the range of speeds ($V^* > 0.38$). Here the fundamental low frequency mode $\omega_{NL2}$ remains, although it is now shifted...
Fig. 13. (a) Bicoherence and (b) tricoherence estimates at $V^* = 0.45$ for the control surface hinge acceleration response.

<table>
<thead>
<tr>
<th>Speed</th>
<th>Linear aeroelastic modes</th>
<th>Nonlinear interactions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>$\omega_h$ 9.77 Hz</td>
<td>$\omega_{NL1} = 2\omega_h + \omega_{NL2} - 2.78$ Hz</td>
</tr>
<tr>
<td></td>
<td>$\omega_x$ 20.08 Hz</td>
<td>$\omega_{NL2} = (\omega_x - 4\omega_h)/2 = 4.04$ Hz</td>
</tr>
<tr>
<td></td>
<td>$\omega_\phi$ 47.15 Hz</td>
<td>$\omega_{NL3} = \omega_{NL4} = 2\omega_h - 6.99$ Hz</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\omega_{NL4} = 4\omega_h - 8.07$ Hz</td>
</tr>
<tr>
<td>0.35</td>
<td>$\omega_h$ 10.16 Hz</td>
<td>$\omega_{NL2} \approx \omega_h/3 = 3.39$ Hz</td>
</tr>
<tr>
<td></td>
<td>$\omega_x$ 19.9 Hz</td>
<td>$\omega_{NL5} = (\omega_y - 2\omega_h)/2 = 3.04$ Hz</td>
</tr>
<tr>
<td></td>
<td>$\omega_\phi$ 47.20 Hz</td>
<td>$\omega_{NL6} = (\omega_y - 2\omega_h)/2 = 3.94$ Hz</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\omega_{NL7} = (\omega_h/2) = 5.37$ Hz</td>
</tr>
<tr>
<td>0.45</td>
<td>$\omega_h$ 10.74 Hz</td>
<td>$\omega_{NL5}$</td>
</tr>
<tr>
<td></td>
<td>$\omega_x$ 19.68 Hz</td>
<td>$\omega_{NL6}$</td>
</tr>
<tr>
<td></td>
<td>$\omega_\phi$ 47.24 Hz</td>
<td>$\omega_{NL7}$</td>
</tr>
</tbody>
</table>

Fig. 14. Powerspectral density estimates of each mode with the system in limit cycle.
to 3.04 Hz and no longer locks into the subharmonic \( \omega_h/3 \). Thus, the system is again tri-modal and aperiodic. Further, \( \omega_{NL6} \) is formed according to the nonlinear interaction \( (\omega_p - 2\omega_h)/2 = 3.94 \) Hz. Weak quadratic phase-coupling is observed (Fig. 13a) where the strongest interactions are between \([(\omega_h),(\omega_h)], [(\omega_h),(2\omega_h)], [(\omega_h),(2\omega_h + \omega_{NL5} = \omega_h + \omega_p - \omega_p/2)] = 24.52 \)

![Fig. 15. Peak-to-peak plots of the LCO region comparing responses obtained via linearized to nonlinear aerodynamic mechanisms.](image1)

![Fig. 16. Pressure coefficient distributions on the airfoil surface at the peak of the cycle for a) \( V' = 0.55 \), b) \( V' = 0.6 \) and c) \( V' = 0.64 \).](image2)
Fig. 17. Maximum bicoherence values in the pitching and control hinge DOFs as a function of velocity index.

Fig. 18. Comparison of responses obtained via linearized and nonlinear aerodynamics at $V/V_\infty = 0.55$ – a) time series, b) phase portrait and c) pitching moment-displacement diagram.
Hz] and \((\omega_h, 2\omega_h + \omega_{NL5} = 2\omega_h + \omega_{NL5}/2 - \omega_y) = 25.42\text{ Hz}\). A moderate interaction via the bi-interaction \([(2\omega_h + \omega_{NL5} = 2\omega_h + \omega_{NL5}/2 - \omega_y, 2\omega_h + \omega_{NL5} = 2\omega_h + \omega_{NL5}/2 - \omega_y)\). Moderate to weak interactions occur at \([(3\omega_h, 3\omega_h + \omega_{NL5} = \omega_h + \omega_{NL5}/2 - \omega_y/2), (3\omega_h, 3\omega_h + \omega_{NL5} = 2\omega_h + \omega_{NL5}/2 - \omega_y)], \[(\omega_h, \omega_h, 2\omega_h + \omega_{NL5} = 2\omega_h + \omega_{NL5}/2 - \omega_y/2)\] and \[(\omega_h, \omega_h, 2\omega_h + \omega_{NL5} = \omega_h + \omega_{NL5}/2 - \omega_y/2)\]. Here, strong cubic nonlinearities are identified in all DOFs. In the control hinge (Fig. 13 b) there is strong cubic phase-coupling via the tri-interaction of \([(\omega_h, \omega_h, \omega_h), (\omega_h, \omega_h, \omega_h)\]) and moderate cubic phase-coupling via the tri-interactions \([(\omega_h, \omega_h, \omega_h), (\omega_h, \omega_h, \omega_h), (\omega_h, \omega_h, \omega_h)]\). It should be noted that at \(V = 0.45\) the resolution of the bi- and tricoherence plots (Fig. 13) make it difficult to distinguish between the interactions \([2\omega_h + \omega_{NL5} = \omega_h + \omega_{NL5}/2 - \omega_y/2 = 24.52\text{ Hz}]\) and \([\omega_h, \omega_h, \omega_h = 2\omega_h + \omega_{NL5}/2 - \omega_y] = 25.42\text{ Hz}\). As opposed to two separate peaks they are displayed as one smeared peak covering the range 24.52–25.42 Hz. The clear separate peaks are visible in the PSD plots (Fig. 10) where a higher resolution is obtained without formidable computational cost.

### 4.4. Limit cycle

In the previous section it is found that at each \(V\) value there is good agreement between the responses obtained using linearized and nonlinear aerodynamic models, which is pertinent to the range \(0.2 < V < 0.5\). Thus, the aerodynamic mechanism can be considered linear within this range of speeds, which is intuitive considering the low amplitude oscillations. Fig. 9 demonstrates that a subcritical bifurcation occurs at \(V = 0.57\) beyond which the dynamics of the system is characterized by high amplitude limit cycle behavior for which the assumption of linearized aerodynamics may no longer be valid.

![Comparison of responses obtained via linearized and nonlinear aerodynamics at \(V = 0.6\).](image-url)
Fig. 14 presents the PSD estimates at $V' = 0.6$. It can be seen that all modes are locked into $\omega_h$, which drives the LCO. This behavior is pertinent to the entire LCO region, and similarly flutter occurs via the same mechanism.

Fig. 15 compares the maxima of the airfoil a) and control hinge b) pitching rotation in the LCO region obtained using the linearized and nonlinear aerodynamic models. In both DOFs a discrepancy between the two solutions becomes apparent at $V' = 0.6$ beyond which the linearized solver predicts exponential growth of the limit cycle amplitude whereas the nonlinear solver predicts slower growth. Essentially, this suggests that the nonlinear aerodynamic contributions have a damping effect on the limit cycle. To investigate this further the pressure coefficients on the upper and lower surfaces of the airfoil at the peak of the LCO trajectory (identical to the trough, however, on opposing sides) are plotted in Fig. 16. Fig. 16 a) ($V' = 0.55$) indicates that pre-LCO a very weak shock is moving back and forth (linear Type-A shock motion) whereas Fig. 16 c) ($V' = 0.64$) indicates that at the peaks and troughs of the cycle the distance that the shock travels is much greater and on the opposing surface it completely disappears (nonlinear Type-B shock motion). Fig. 16 b) ($V' = 0.6$) demonstrates that as the airfoil moves to the peak or trough of its trajectory the strength of the shock can be seen to increase and move downstream on one surface while on the opposing surface it becomes extremely weak (nearly disappearing completely) and moves upstream. Although this is not characteristic of what is typically considered Type-B shock motion, it is still inherently a nonlinear aerodynamic phenomenon. Fig. 17 compares the maximum bicoherence estimates of the airfoil and control hinge pitching rotation obtained using the linearized and nonlinear aerodynamic models. It can be seen that the bicoherence estimates of the airfoil rotation obtained using the nonlinear aerodynamic model grow as the system approaches LCO becoming near unity at $V' = 0.59$, conversely the control hinge rotation is consistently characterized by moderate-weak quadratic phase-coupling. In both of these DOFs the response obtained via the linearized aerodynamic model is consistently

![Graphs showing time series, phase portrait, and pitching moment-displacement diagram](image_url)
characterized by weak quadratic phase-coupling. These findings suggest that the nonlinear shock motion is characterized by a quadratic form and that the strength of the aerodynamic nonlinearity can be quantified by the magnitude of the bicoherence (the strength of the quadratic phase-coupling between modes).

It has been identified that nonlinear shock motion, characterized by a quadratic process, is inherent within the system as the amplitude of the LCO grows. The resultant 3DOF aeroelastic system now contains multiple types of nonlinearity simultaneously, which must be considered in the identification process. To investigate the interaction between the structural and aerodynamic nonlinearities, Figs. 18–20 present time-series, phase-portrait and displacement-moment plots comparing the responses obtained using the linearized to nonlinear aerodynamic model. Fig. 18 demonstrates that pre-LCO there is good agreement between the aerodynamic models for both the aperiodic control surface and the quasi-periodic airfoil rotation responses. Fig. 19a) and b) indicate that the nonlinear aerodynamic mechanism on the airfoil body is reducing the amplitude of the airfoil and to a lesser degree the control hinge rotation, however, the general form for both DOFs appears to remains consistent, i.e., the nonlinear shock motion is having a scaling effect. Fig. 19c) indicates a significant difference in the aerodynamic mechanisms when comparing the linearized to nonlinear aerodynamic models at this speed. The linearized aerodynamic model indicates that the trajectory of the pitching moment as the airfoil rotates from peak to trough is of a linear form with a cubic influence (as to be expected due to the cubic freeplay mechanism), however, in terms of the nonlinear aerodynamic response a sharp (almost instantaneous) drop in pitch occurs at the peaks and troughs of the cycle as a result of the appearing/disappearing shock on the opposed surface. Thus, the trajectory of the pitching moment as the airfoil rotates from peak to trough is now characterized by a clear quadratic form, which gives insight towards the identified quadratic nonlinearity in the pitch response. Similar observations can be made in Fig. 20, however, amplified due to the increased strength of the aerodynamic nonlinearity.

Fig. 21. Higher-order frequency content for the airfoil pitching response at \( V' = 0.6 \): a) bicoherence linearized aerodynamics, b) bicoherence nonlinear aerodynamics, c) tricoherence linearized aerodynamics and d) tricoherence nonlinear aerodynamics.
Finally, Figs. 21 and 22 present HOS estimates comparing the responses obtained using the linearized to nonlinear aero-
dynamic model at $V^*/C_3 = 0.6$. This allows for further investigation into the interaction between the structural and aerodynamic nonlinear mechanisms. It can be seen that the airfoil and control hinge pitching responses are characterized by strong cubic phase-coupling via the tri-interaction of $\omega_h$, and the presence of the aerodynamic nonlinearity has no impact on the identification of the cubic process (i.e., both the phase and magnitude of the cubic phase-coupling are not impacted in the presence of the aerodynamic nonlinearity). When comparing the bicoherence estimations between the two aerodynamic models, consistency is observed with respect to the identification of frequencies, however, significant discrepancies can be observed in the magnitude. This is consistent for all speeds post-LCO.

To summarize, for the present 3DOF system, from a physical perspective the nonlinear shock motion does not appear to impact the form of the response (defined by the structural freeplay), however, does have a damping effect, i.e., the amplitude of the oscillations in all DOFs is reduced. From a system identification perspective it does not impact the frequency interactions associated with the quadratic phase-coupling but does increase the magnitude, i.e., the inviscid aerodynamic nonlinearity is characterized by a quadratic process and can be identified by the presence of quadratic phase-coupling between linear and nonlinear modular modes. Furthermore, the magnitude and frequency interactions associated with the cubic phase-coupling appears to be unchanged and hence for the present 3DOF system it can be said that the aerodynamic nonlinearity does not impact the identification of the freeplay.

### 4.5. Discussion

The findings provided in the current section suggest that the control hinge freeplay nonlinearity is characterized by a cubic process. At lower speeds (pre-limit cycle) the low amplitude dynamic response is driven by the structural freeplay.
and can be characterized by aperiodic, quasi-periodic and periodic behaviors. Furthermore, nonlinear modes can be identified as defined by interactions between the linear aeroelastic modes. All nonlinear interactions are characterized by the presence of the freeplay nonlinearity, which is a promising finding in consideration of freeplay source detection and is to be further investigated in future work. The composition of the nonlinear modes (i.e., the ratio of natural linear frequencies which define them) changes as a function of speed. It is evident that the periodicity of the system is dependent on the number of effective modes acting, i.e., at lower speeds when the three modes are uncoupled the behavior is aperiodic. However, as the speed increases and two modes become harmonically related the system effectively becomes bimodal and quasi-periodic/periodic behavior is observed. At higher speeds a subcritical bifurcation occurs and the system exhibits highly periodic limit cycle flutter behavior which is driven predominantly by the transonic aerodynamic forces with all DOFs locked into the plunging mode. The response obtained using the linearized aerodynamic model indicates exponential growth of the LCO until catastrophic failure at the linear flutter speed. However, the nonlinear aerodynamic model indicates that as the amplitude of the limit cycle grows the nonlinear inviscid aerodynamics takes effect and can be characterized by a quadratic process. It is evident that the nonlinear shock motion dampens the limit cycle response and modest growth of the amplitude is observed. Although the nonlinear shock motion dampens the response, it does not change the general form of the trajectory. This indicates that the presence of the quadratic inviscid aerodynamic process does not interact significantly with the cubic characteristics of the freeplay mechanism and hence identification of the freeplay nonlinearity remains unchanged. Although the nonlinear frequencies are dependent on interactions between all linear aeroelastic modes, the linear aeroelastic modes themselves are difficult to identify; most notably and its harmonics. This is addressed in Part I of the paper.

5. Conclusion

The higher-order spectral content of a 3DOF airfoil system in transonic flow with control surface freeplay is investigated with two aerodynamic models, namely linearized and nonlinear inviscid aerodynamics. By utilizing the linearized aerodynamic model, the higher-order spectral content of a freeplay nonlinearity in isolation is quantified. Then, by incorporating the inviscid nonlinear aerodynamic solver and comparing with the solutions as generated by the linearized aerodynamic solver, the effects of linear, weakly nonlinear and nonlinear transonic flow are also discussed.

Structural freeplay nonlinearity can be characterized by cubic phase-coupling between linear and nonlinear modular modes. On the other hand weak-moderate aerodynamic nonlinearities due to inviscid shock motion can be characterized by quadratic phase-coupling between the linear and nonlinear modular modes and the magnitude of the quadratic phase-coupling is found to be proportional to the magnitude of the nonlinear shock motion. The freeplay phenomenon is characterized by complex modular frequency interactions where all three linear aeroelastic modes (and its harmonics) interact to formulate a set of nonlinear modes. The presence of aerodynamic nonlinearity does not appear to impact the structural freeplay mechanism, nor does it affect freeplay identification.

The periodicity of the system prior to nonlinear flutter appears to be defined by the number of effective modes acting within the nonlinear process. For example when all three linear aeroelastic modes are non-harmonically related the response appears aperiodic, however, if two modes become harmonically related the number of effective modes is reduced and inherently the response shifts to a higher state of order (quasi-periodic/periodic behavior). It is found that the quasi-periodic behavior continues for a finite range of speeds despite the harmonic relationship between modes no longer existing. Eventually, a breakdown of the quasi-periodicity is observed and the system returns to disorder. Conjectures towards a physical explanation for these phenomena are discussed in this Part I, and further investigated/supported in Part II.

For all reduced velocity values the magnitude of the tricoherence is the largest at the control surface (although this becomes marginal at greater speeds). This seeds future work focusing on HOS and their ability to provide insight into the spatial source of the nonlinearity in industrial scale SHM problems. In order to do so, a rigorous study is required into the number of time-steps necessary to estimate the HOS with sufficient resolution while maximizing computational efficiency.

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Appendix A. Time-series and block length convergence

Using the example case presented in Section 4.2, a time-series and block length convergence study is conducted. As previously mentioned, the recommended time-series length of Dalle Molle and Hinch leads to a formidable computational bur-
den. This convergence study aims to identify a reasonable trade-off between the accuracy of the HOS value estimate and the computational cost of obtaining the solution.

Figs. A.23 and A.24 a) and b) indicate that there is an exponential relationship between the number of time-steps and the total CPU time taken to obtain a HOS estimation. Hence, reduction of the time-series length is essential. This is less of a concern when considering the bicoherence estimates with 1024-point FFT using a total of 1024/512 = 524.288 data points (A.23a) and (b) where the total time is approximately \( t = 575 \) s or 9.6 min. However, it becomes extremely evident when estimating the tricoherence with a 1024-point FFT using a total of 1024/512 = 524.288 data points (A.24a) and b) for which the total time to obtain an estimation is \( t = 31.932 \) s or 8.9 h.

Fig. A.23 c) shows that when using a 512-point FFT there is a significant amount of variance in the tricoherence estimate and for convergence no less than 512 × 128 = 65,536 data points should be used as a minimum. Whereas, Fig. A.23 d) indicates that when using a 1024-point FFT no less than 1024 × 32 = 32,768 data points should be used.

Fig. A.24 c) shows that when using a 512-point FFT there is a significant amount of variance in the tricoherence estimate and for convergence, no less than 1024 × 128 = 131,072 data points should be used as a minimum. Whereas, Fig. A.24d) indicates that when using a 1024-point FFT no less than 1024 × 32 = 32,768 data points should be used.
Appendix B. Spatial and temporal convergence

In the present research, spatial and temporal resolution of both the structural and aerodynamic mechanisms are essential to ensure that the physics is captured with a reasonable balance between the fidelity and computational cost. Studies are conducted for the most demanding scenario in each mechanism.

Temporal resolution is essential predominantly from a structural perspective at low speeds. This is because at low speeds the control surface behaves chaotically and can be characterized by interactions between all three structural modes, and these complex interactions have been termed “nonlinear modes”. As is presented in Section 4.3, the frequencies which are required to capture the linear and nonlinear modes are approximately within range of 1–70 Hz. Hence, the temporal resolution study is of particular importance here to gain confidence in the temporal fidelity being sufficient to capture the complex high- and low-frequency interactions.

Spatial resolution is most important at high speeds when the amplitude of the airfoil pitching motion induces a transition between Type-A and Type-B shock-motion. Evidently this complex aerodynamic phenomenon requires a computational grid with a spatial resolution of a fidelity fine enough to capture the properties of the nonlinear shock motion, e.g., the mean shock location, the maximum forward and aft locations of the shock motion, and the velocity at which the shock travels.

Fig. A.24. Computational time as a function of time-series length with a) the x-axis in logarithmic scale and b) both the x-axis and y-axis in logarithmic scale, and maximum tricoherence as a function of time-series using c) a 512-point FFT and d) a 1024-point FFT.

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B.1. Spatial resolutions

The maximum speed considered is $V' = 0.64$ (just below the linear flutter boundary) where the system is in limit cycle, this is the most extreme condition (from an aerodynamic perspective), i.e., the airfoil experiences maximum pitching displacement and inherently the shock displacement is largest. As was previously mentioned, there are many parameters related to the nonlinear shock motion which need to be considered in this spatial resolution study. In this study, at a speed of $V' = 0.64$, four parameters are chosen to study the nonlinear aerodynamics; i) the maximum forward; and ii) aft shock locations for the steady-state response at the peak of the cycle, and iii) the frequency; and iv) magnitude of the most prominent peak in the bicoherence estimation for the steady-state time-history of the pitching moment on the main airfoil section. The bicoherence is used as a parameter as the nonlinear shock motion is defined by a quadratic process as presented in Section 4.4.

As this paper presents time-marching Euler solutions to the aerodynamic forces, the requirements of the spatial resolution are not as strict as they would be if a turbulence closure were to be taken into account. The use of Euler aerodynamics is considered valid in this work as the maximum pitching amplitude of the airfoil at $V' = 0.64$ induces inviscid nonlinear phenomena only, i.e., viscous effects (referred to in the introduction) do not need to be taken into account. Although not shown here, at the upper and lower speed extremities ($V' = 0.25$ and 0.64) the time-marching structural response in each DOF produced by transient Euler-CFD is indifferent to that of the transient RANS-CFD solutions to the generalized aerodynamic forces.

The grid convergence index (GCI) of Roache [39] is used to conduct the mesh convergence study for three levels of mesh resolution. The GCI provides an error band for the range of resolution that is considered. Provided that uniform refinement is concerned, the discretized solution will approach the actual solution (infinitely fine, i.e., zero grid resolution) which is verified by ensuring that the solutions are within the asymptotic range of convergence.

To calculate the GCI, initially the order of convergence $P$ is calculated according to

$$P = \frac{\ln|j_3 - j_2|/|j_2 - j_1|}{\ln(2)}$$

where $j_1$, $j_2$, and $j_3$ are the solutions calculated from three grids for which uniform refinement is considered in ascending order, i.e., $j_1$ represents the solution calculated for the coarse grid and $j_3$ for the fine grid. The theoretical value of $P$ is 2 and any deviation can be attributed to grid quality as well as nonlinearities associated with the structure, aerodynamic and thermodynamic phenomena, or turbulent properties (although not applicable in the present work), etc.

The GCI can now be calculated while considering a safety factor. As the grid refinement is uniform and two levels of refinement are considered (three grids in total) a safety factor of $F_s = 1.25$ is used. The GCI for grids 1–2 and 2–3 are now calculated according to

$$GCI_{12} = \frac{F_s|j_1 - j_2|/|j_2 - j_1|}{(2^P - 1)}$$

$$GCI_{23} = \frac{F_s|j_2 - j_3|/|j_3 - j_2|}{(2^P - 1)}$$

Finally, to ensure that the solutions are within the asymptotic range of convergence the following equation is used

$$A_r = \frac{GCI_{23}}{GCI_{12}^2}$$

where, $A_r$ represents the asymptotic range, which should have a value of approximately one to indicate asymptotic convergence.

Table B.3 presents the attributes of the grids considered and their respective solutions to the four parameters. A uniform refinement is considered such that for the medium resolution the mesh density in both axial and radial directions is increased by a factor of 2, and for the fine resolution the mesh density in both radial and axial directions is increased by a factor of 4 from the coarse resolution.

Table B.4 presents a summary of the grid convergence index which is calculated for the parameters, as $x_{for}$ is consistently zero, i.e., the shock completely disappears on the lower surface at the peak of the cycle, this is not included in the GCI cal-

<table>
<thead>
<tr>
<th>Grid</th>
<th>Refinement level</th>
<th>$n_{pol}$</th>
<th>$x_{for}$</th>
<th>$x_{gt}$</th>
<th>$f_{b,\text{max}}$ (Hz)</th>
<th>$h_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coarse(1)</td>
<td>1</td>
<td>(120 × 46)</td>
<td>0</td>
<td>0.653</td>
<td>12.23</td>
<td>0.923</td>
</tr>
<tr>
<td>Nominal(2)</td>
<td>2</td>
<td>(240 × 92)</td>
<td>0</td>
<td>0.662</td>
<td>12.18</td>
<td>0.928</td>
</tr>
<tr>
<td>Fine(3)</td>
<td>4</td>
<td>(480 × 184)</td>
<td>0</td>
<td>0.669</td>
<td>12.05</td>
<td>0.935</td>
</tr>
</tbody>
</table>
Parameters and results of the temporal resolution study.

Table B.4

<table>
<thead>
<tr>
<th>Grid step</th>
<th>Refinement ratio</th>
<th>$x_{\text{eff}}$ (GCI (%))</th>
<th>$f_{b,\text{max}}$ (GCI (%))</th>
<th>$b_{\text{max}}$ (GCI (%))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–2</td>
<td>2</td>
<td>2.30</td>
<td>2.50</td>
<td>1.69</td>
</tr>
<tr>
<td>2–3</td>
<td>2</td>
<td>1.14</td>
<td>3.20</td>
<td>2.36</td>
</tr>
</tbody>
</table>

Calculations. For all parameters the coarse grid is sufficient with an error band of 2.30% for $x_{\text{eff}}$, 2.50% for $f_{b,\text{max}}$ and 1.69% for $b_{\text{max}}$. Furthermore, the asymptotic range for all parameters is approximately at unity and hence are well within the asymptotic range of convergence.

B.2. Temporal resolution

As previously discussed the temporal resolution is important at low-speeds where the time-step should be low enough to capture a range of nonlinear modes which have been identified, i.e., those which have no harmonic relation to the linear aeroelastic modes, but rather, are characterized by complex interactions which occur between them. $V^* = 0.25$ is considered to study the temporal resolution as this is the lowest speed to be considered in the study, the response is chaotic and several complex nonlinear modes have been defined (presented in Section 4.3). Furthermore, at this speed the amplitude of the structural response in each DOF is low, meaning that the only nonlinearity acting upon the system is the structural freeplay in the control surface, i.e., aerodynamic nonlinearities are not present (this is shown and further examined in Sections 4.3 and 4.4). At this speed, the structural freeplay nonlinearity is now considered to be isolated, which is assumed with certainty if the aerodynamic forces are calculated using the linearized solver, or with great confidence if the nonlinear time-marching solver is used. Hence, the nonlinear phenomena observed at this speed are assumed with confidence to be an artifact of the structural freeplay in the control surface only. It follows that the control surface pitching response is used to investigate the temporal resolution.

Four time-steps are considered each with a total of $N = 2^{17}$ data points for the linear and nonlinear spectral estimates as defined in Table B.5. Due to the chaotic nature of the response and the vast range of frequencies which are present, it is difficult to define a single parameter to quantify the fidelity of the response as the time-step varies. The main focus of the paper at low speeds is to investigate the presence and composition of nonlinear modes in the chaotic/aperiodic control surface rotational response. Furthermore, it is shown in Section 4.3 that at $V^* = 0.25$ the temporal fidelity must be sufficient to capture frequencies as low as approximately 2.5 Hz in this Part I and approximately 1.5 Hz in Part II [1]. Considering these objectives, the temporal convergence study will compare the frequencies of the nonlinear modes which are identified at $V^* = 0.25$ via the PSD estimation of the control surface hinge rotation response.

The temporal convergence study includes the variance in maximum bi- and tricoherence estimations in all DOFs. This is essential from a system identification viewpoint as one of the objectives of this research is to investigate the use of nonlinear spectral estimations to determine the type of nonlinearity and where in the system the nonlinearity is acting, more detail on which is provided in the Introduction.

For the time-step to be acceptable, the percentage difference from the lowest time-step, $\Delta t = 0.0005$ s, should be no greater than 5%. The results presented in Table B.6 indicate that for $\Delta t = 0.001$ s, the difference in the frequencies of the nonlinear modes and the maximum tricoherence estimate are well within the acceptable range. Although the difference in maximum bicoherence estimate at $\Delta t = 0.001$ s appears to be unacceptable at −5.56%, inspection of the values of this parameter, i.e., 0.19 and 0.18 at $\Delta t = 0.0005$ s and 0.001 s respectfully, suggests that $\Delta t = 0.001$ s is actually sufficient to capture the maximum bicoherence. As the time-step increases to $\Delta t = 0.005$ s and 0.01 s the values of all parameters are well outside the acceptable range. Furthermore, as expected, it can be seen that the percentage difference decreases as the frequencies of the nonlinear modes increases. As is indicated in Reference [31] and confirmed in the present paper (Sections 4.3 and 4.4) at low speeds when the freeplay nonlinearity acting in unison, the system is characterized by cubic phase-coupling only (high tricoherence value) with a very low or negligible bicoherence value. Hence, for time-steps greater than $\Delta t = 0.001$ s the

Table B.5

<table>
<thead>
<tr>
<th>$\Delta t$ (s)</th>
<th>$t_{\text{total}}$ (s)</th>
<th>$f_{NL1}$ (Hz)</th>
<th>$f_{NL2}$ (Hz)</th>
<th>$f_{NL3}$ (Hz)</th>
<th>$f_{NL4}$ (Hz)</th>
<th>$b_{\text{max}}$</th>
<th>$t_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0005</td>
<td>32.768</td>
<td>2.69</td>
<td>3.90</td>
<td>7.08</td>
<td>8.06</td>
<td>0.19</td>
<td>0.665</td>
</tr>
<tr>
<td>0.001</td>
<td>65.536</td>
<td>2.78</td>
<td>4.04</td>
<td>6.99</td>
<td>8.07</td>
<td>0.18</td>
<td>0.875</td>
</tr>
<tr>
<td>0.005</td>
<td>327.68</td>
<td>3.40</td>
<td>4.88</td>
<td>6.42</td>
<td>8.30</td>
<td>0.3</td>
<td>0.976</td>
</tr>
<tr>
<td>0.01</td>
<td>655.36</td>
<td>3.7</td>
<td>4.88</td>
<td>6.35</td>
<td>9.01</td>
<td>0.47</td>
<td>0.270</td>
</tr>
</tbody>
</table>
nonlinearities in the system are misrepresented. With all of these findings in consideration, the present work proceeds with a time-step of $\Delta t = 0.001$ s.

References

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